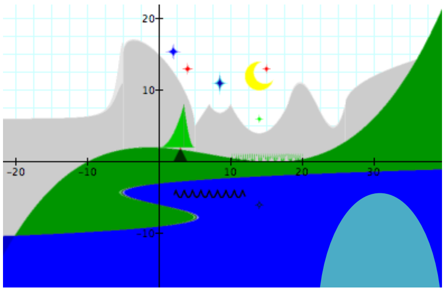
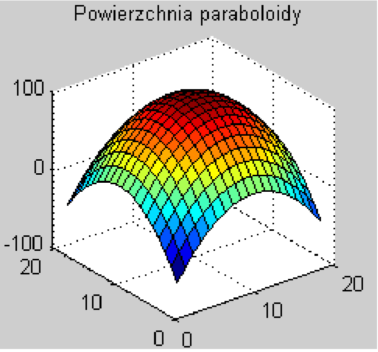
Algebraic Functions

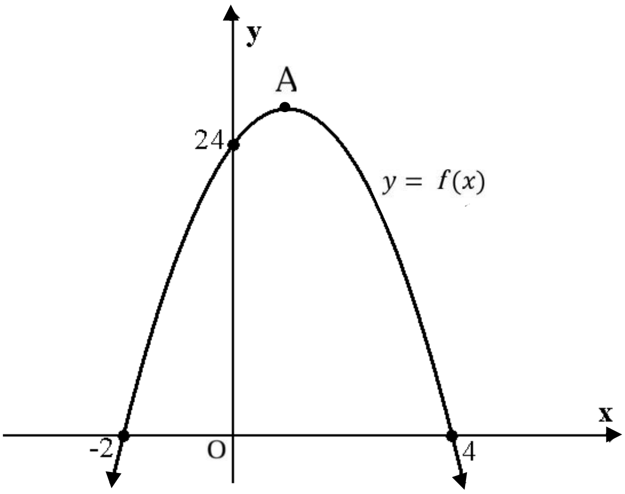




To excel in this topic, a thorough grasp of the following is essential:

- The 4 algebraic functions, viz. the straight line $y = mx + c$, the parabola $y = ax^2 + bx + c$, the exponential function $y = a.b^x + q$ and the hyperbola $y = \frac{k}{x-p} + q$. The cubic function $y = ax^3 + bx^2 + cx + d$ will be treated under the topic *Differential Calculus*.
- The *shape* of each of the above and their inverses.
- The influence of the parameters, a, b, c, p, q, etc. on each of the above functions.
- Where applicable, their x-and y-intercepts, vertices and asymptotes.
- The influence of shifts, vertical and/or horizontal and also reflections in the x-axis, the y-axis and in the $y = x$ and $y = -x$ lines on the *equations* and *graphs* of these five functions.

1. The diagram shows the graph of $y = f(x)$.



1.1 Determine the equation of this graph in the form $y = a(x-\alpha)(x-\beta)$

1.2 Write the equation of f in the form $y = a(x-p)^2 + q$ by completing the square.

1.3 Hence write down the coordinates of A, the vertex of the graph of f.

1.4 Explain how you would shift the graph of f so that it will have two equal roots.

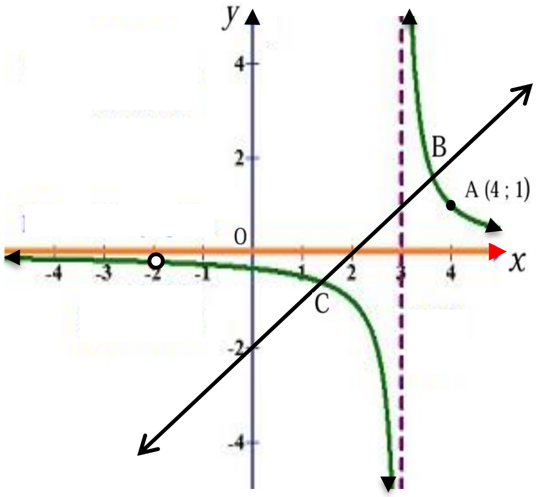
1.5 If $p(x)$ is the reflection of the graph of $f(x)$ in the y-axis, write down the equation of $p(x)$.

2. Make neat sketches of the function $y = ax^2 + bx + c$, each on its own system of axes for:

2.1 $y = f(x)$ with $a < 0$, $b > 0$ and $b^2 - 4ac = 0$

2.2 $y = g(x)$ with $a > 0$, $b < 0$ and $b^2 - 4ac < 0$

3. The sketch on the right represents the graphs of the functions whose equations are defined by $f(x) = \frac{a}{x-p} + q$, $x \neq -2$ and $g(x) = mx + c$ respectively.



3.1 Determine the values of a, p and q for the graph of f and the values of m and c for that of g.

3.2 Give the domain of f and hence write down the equation for $f(x)$.

3.3 Determine the coordinates of B and C, the points of intersection of f and g.

3.4 Determine the equations of the axes of symmetry for the graph of $f(x)$.

3.5 Use the graph and write down the values of x for which:

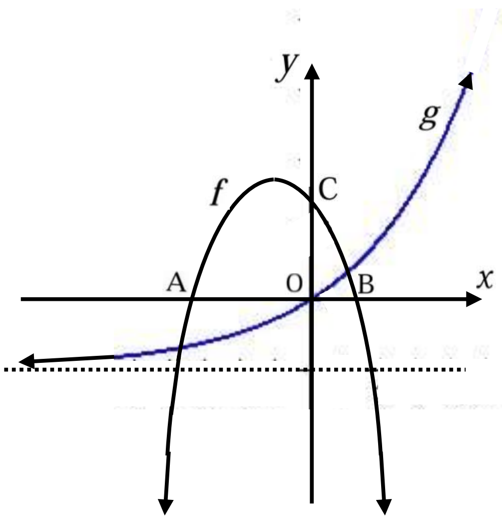
3.5.1 $f(x) \geq g(x)$
3.5.2 $f(x).g(x) \leq 0$
3.5.3 $x.g(x) < 0$

4.1 Draw the graph of $f(x) = 2^{x+1} - 8$. Indicate clearly all intercepts with the axes as well as the asymptote.

4.2 The graph of $g(x)$ is obtained by reflecting f in the y-axis. Write down the equation of g.

4.3 If the two graphs $f(x)$ and $g(x)$ intersect each other at Q, write down the coordinates of Q.

5. The sketch represents the graphs of the functions $f(x) = ax^2 + bx + c$ and $g(x) = d^x + q$
The x-intercepts of f are A$(-3; 0)$ and B$(1; 0)$.
The y-intercept of f is C$(0; 6)$. The graph of g passes through the origin and the point $(1; 2)$.



5.1 Determine the values of d and q.

5.2 Determine the values of a, b and c.

5.3 Use the graph to determine the values of x where $f'(x).g(x) < 0$

5.4 Determine the equation of g^{-1}, the inverse of g in the form $y = \ldots..$

5.5 State the domain of g^{-1}.

5.6 Determine the values of k for which $f(x) + k = g(x)$ will have two real roots that are opposite in sign.

5.7 Determine the equation of f^{-1} in the form $y = \ldots.$ and hence sketch the graph of f^{-1}, showing the coordinates of its turning point (vertex) and the intercepts with the axes.

6. This question is a follow up on question 5 of this section and should preferably be attempted only after answering question 5.

6.1 Are the inverses of all functions also functions? Explain your answer with reference to the two inverse functions g^{-1} en f^{-1} (of question 5).

6.2 Explain, and again with reference to the functions in question 5, how one can, in two ways, change or restrict the domain of a given function whose inverse is not a function for it to be a function.

6.3 On two separate system of axes, illustrate the graphs of the inverses of the functions f^{-1}, giving the equation of each.

7.1 The equation of a linear function is defined as $g(x) = 2x + 1,\ x \in [-3; 2)$.
Determine the equation of g^{-1}, the inverse of g in the form $y = mx + c$.

7.2 Determine the values of x and y for which $g(x) = g^{-1}(x)$. Solve, in other words, simultaneously for x and y.

7.3.1 Sketch the graphs of g and its inverse g^{-1}, both on the same system of axes and for $x \in R$ for both functions, indicating clearly the x- and y- values of the intercepts with the axes as well as with the $y = x$ line.

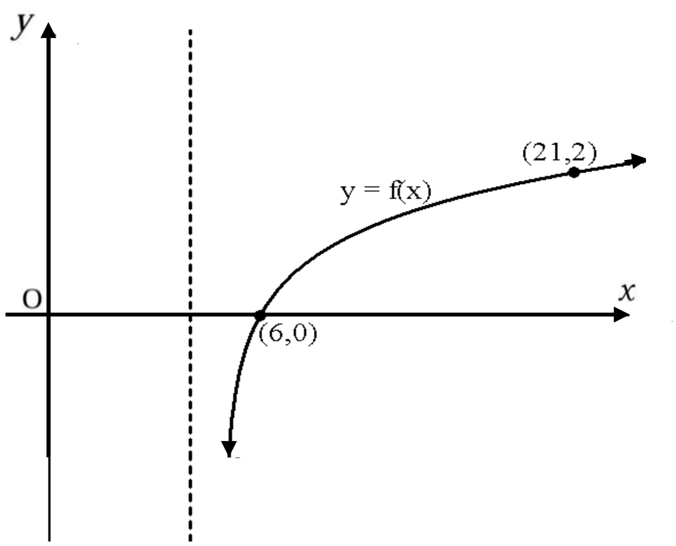
7.3.2 Repeat question 6.3.1 on a different system of axes, but sketch the two functions for the domain of g as given initially, viz, $x \in [-3; 2)$.

8.1 The diagram shows the graph of $y = f(x) = \log_b(x + q)$.
Determine the equation of $f(x)$.

8.2 Determine the values of x for which $f(x) \leq -2$.

8.3 Determine the equation of f^{-1}, the inverse of f.

8.4 Sketch the graph of f^{-1}, indicating clearly the values (or coordinates) of any intercepts with the axes that the graph may have and also indicate the the asymptote's equation.

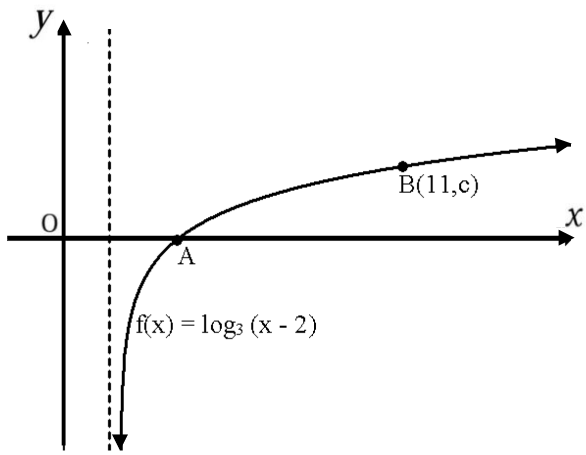


9.1 The diagram shows the graph of $f(x) = \log_3(x-2)$. The point B (11 ; c) lies on this graph. Determine the value of c.

9.2 Write down the equation of the asymptote of f.

9.3 Determine the values of x for which $f(x) < 2$?

9.4 If the graph of f is reflected in the x-axis , determine the equation of the reflected graph.



10. Given: $h(x) = 2x - 3,\ -2 < x \leq 4,\ x \in R.$

10.1 Determine the coordinates of Q.

10.2 Give the equation of h^{-1} in the form $y = \ldots$ stating the function's domain.

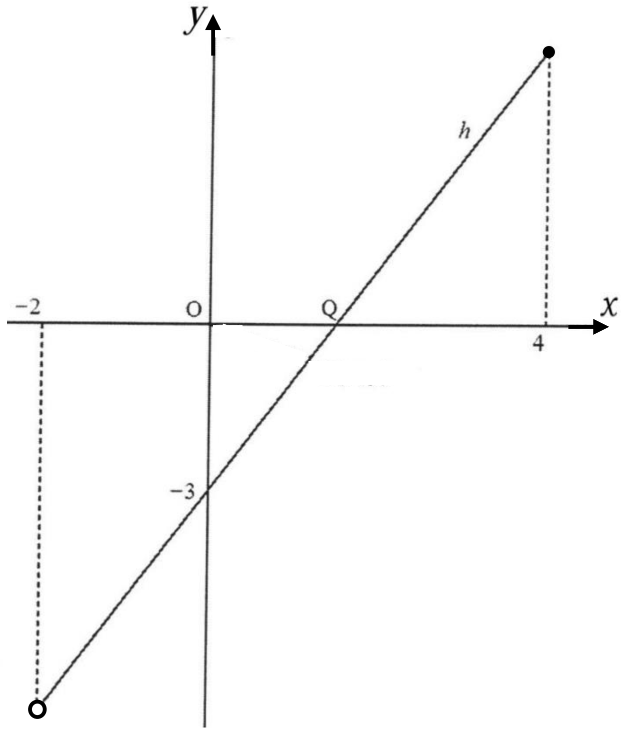
10.3 Sketch the graph of h^{-1}, clearly indicating the x- and y-intercepts and the coordinates of the end points.

10.4 For which value(s) of x will $h(x) = h^{-1}(x)$

10.5 Given: $h(x) = f'(x)$ where f is a function defined for $-2 < x \leq 4$.

10.5.1 Explain why f as a local minimum.

10.5.2 Write down the maximum gradient of the tangent to the graph of f.



11. The function defined as $y = \dfrac{a}{x+p} + q$ has the following properties:

- The domain is $x \in R,\ x \neq -2.$
- $y = 6$ is an axis of symmetry.
- The function is increasing for $x \in R,\ x \neq -2.$

11.1 Draw a neat sketch graph of this function. Your sketch must include the asymptotes, if any.

11.2 Give the equation of this function's reflection in:

10.2.1 The x-axis.

10.2.2 The y-axis.

10.2.3 The $y = x$ line.

12. The graph of g is defined by the equation $g(x) = \sqrt{ax}$. The point $(8; 4)$ lies on g.

12.1 Determine the value of a.

12.2 If $g(x) > 0$, for which values of x will g be defined?

12.3 Determine the range of g.

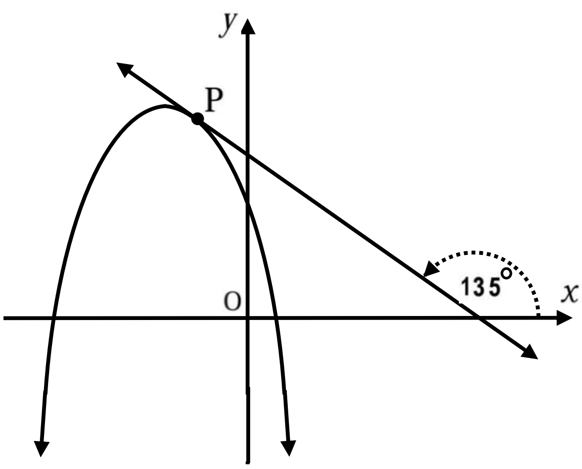
12.4 Give the equation of g^{-1}, the inverse of g in the form $y = \ldots$.

12.5 Sketch the graphs of g and g^{-1} on the same system of axes and write down the value(s) of x for which $g(x) = g^{-1}(x)$.

.12.6.1 If $h(x) = x - 4$ is drawn, determine <u>algebraically</u> the intersection(s) of g and h.

12.6.2 Hence, or otherwise, determine for which values of x is $g(x) > h(x)$.

13. The sketch on the right depicts the graphs of $f(x) = -2x^2 - 5x + 3$ and $g(x) = ax + q$. The graph of g is a tangent to f at P, forming an angle of inclination of 135^0.



13.1 Determine the coordinates of the turning point of the graph of f.

13.2 Determine the coordinates of P, the point of tangency between f and g.

13.3 Hence, or otherwise, determine the equation of the graph of g.

13.4 If Q and N respectively are the x-intercepts of g and its normal, NP, calculate the length of NQ.

13.5 Determine the values of d for which the line $k(x) = -x + d$ will not intersect the graph of f.

14. Given: $f(x) = -x^2 + 2x + 3$ and $g(x) = 1 - 2^x$.

14.1 Sketch the graphs of f and g on the same set of axes.

14.2 Determine the average gradient of f between $x = -3$ and $x = 0$.

14.3 For which value(s) of x is $f(x).g(x) \geq 0$?

14.4 Determine the value of c such that the x-axis will be a tangent to the graph of h, where $h(x) = f(x) + c$.

14.5 Determine the y-intercept of t if $t(x) = -g(x) + 1$.

14.6 The graph of k is a reflection of g about the y-axis. Write down the equation of k.

15. The sketch represents the graphs of $f(x) = \frac{2}{x-3} - 1$ and $g(x) = dx + e$.
Point $B(3; 6)$ lies on the graph of g and the two graphs intersect at points A and C.

15.1 Write down the equations of the asymptotes of f.

15.2 Write down the domain of f.

15.3 Determine the values of d and e correct to the nearest integer if the graph of g makes an angle of 76^0 with the x-axis.

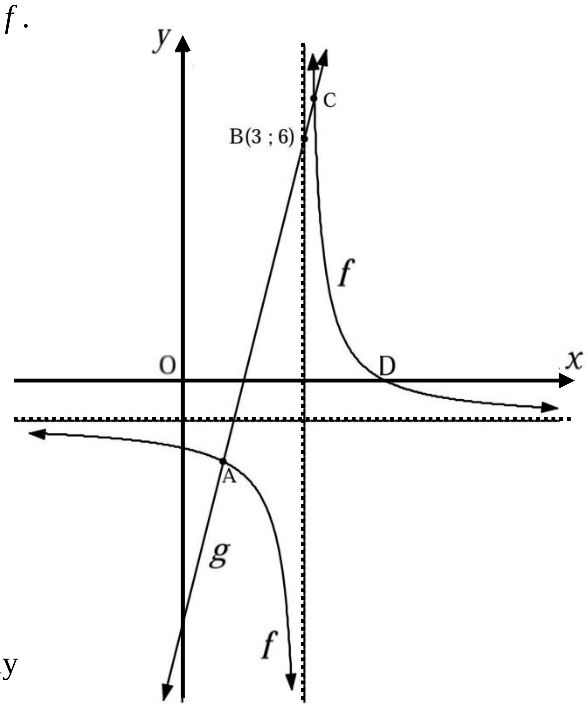
15.4 Determine the coordinates of A and C.

15.5 For what values of x is $g(x) \geq f(x)$?

15.6 Determine an equation for the axis of symmetry of f which has a positive slope.

15.7 If the graph of $p(x)$ is the reflection of f in the $y = x$ line, write down the equations of the asymptotes of $p(x)$.

15.8 Determine the equation of a new function $h(x)$, which results from shifting $f(x)$ horizontally in such a way that the point D is at the origin.



16. The equations of the quadratic functions sketched below are represented in two of the three possible ways. Study the diagrams and answer the questions that follow:

16.1 If $y = a(x-p)(x-q),\ p \leq q$ represents the equation of each of the four graphs illustrated here,

a) In which graph is $p = q$?

b) In which graph is $pq = 0$?

c) In which graph is $a < 0$ and $q > 0$?

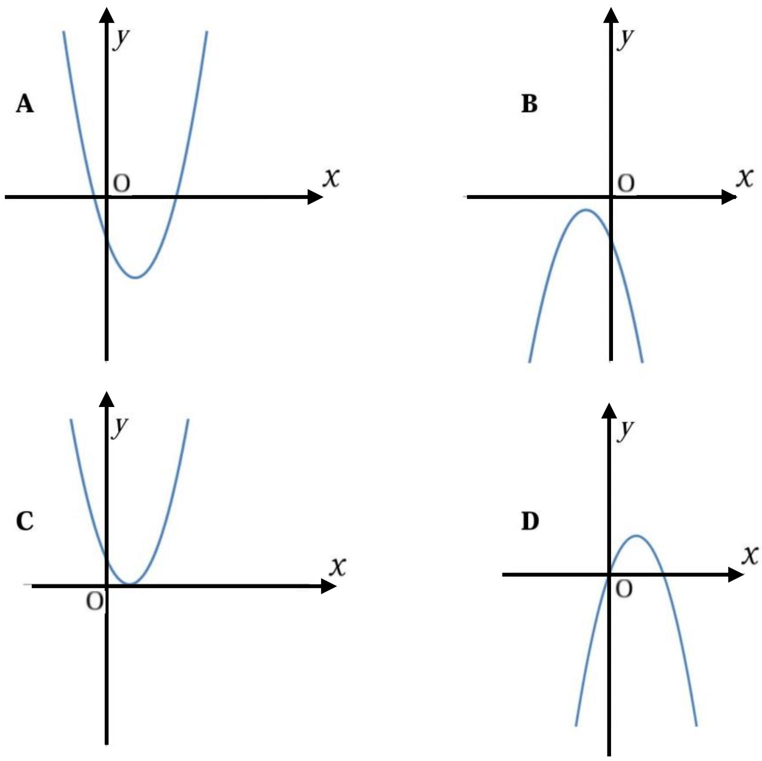
d) In which of the graph(s) is $p + q > 0$?

16.2 If $y = ax^2 + bx + c$ now represents the equation of each of the four graphs illustrated here,

a) In which graph is $a < 0$ and $b < 0$?

b) In which graph is $b^2 - 4ac < 0$?

c) In which graph is $b^2 - 4ac = 0$?



17. The equation of an hyperbola is given as: $f(x) = \frac{x-5}{x+2}$

17.1 Write the equation of f in the standard form, viz: $f(x) = \frac{k}{x-p} + q$

17.2 Write down the equations of the two asymptotes of f.

17.3 Determine the domain of f^{-1}, the inverse of f.

18. The equation of an hyperbola is given as: $f(x) = \frac{2x+1}{x-3}$

18.1 Write the equation of f in the standard form.

18.2 Write down the equations of the two asymptotes of f.

18.3 Determine the domain of f^{-1}, the inverse of f.

19. If $g(x) = \frac{x-6}{x-2}$ and $h(x) = 3^x - 2$ then:

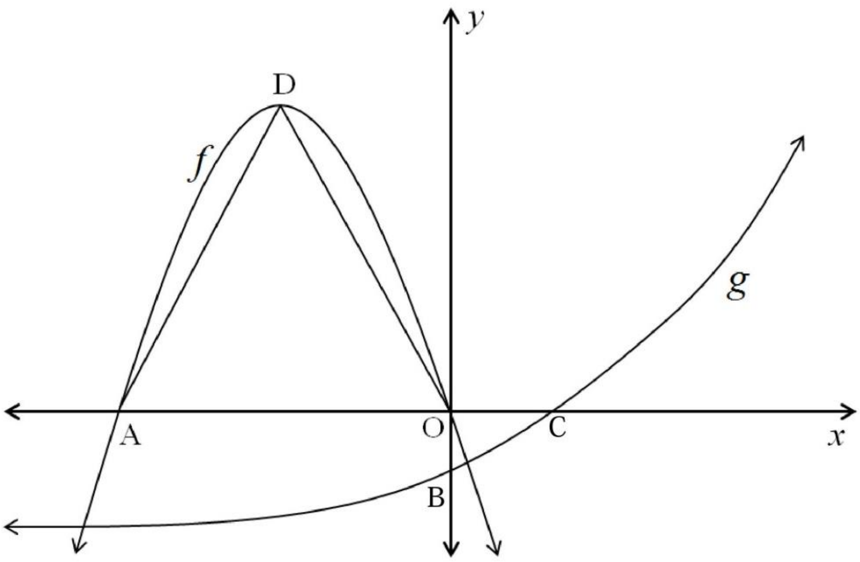
19.1 Determine $h^{-1}(x)$, the inverse of $h(x)$ in the form $y = \ldots$.

19.2 Sketch the graph of h^{-1}.

19.3 For what values of x is $h^{-1}(x) > 0$?

19.4 The graph of $h(x)$ is translated vertically so that it passes through the point of intersection of the asymptotes of $g(x)$. Find the new equation of h.

20. In the diagram below, $f(x) = -x^2 - 4x$ and $g(x) = 2^x - 6$. Point D is the turning point of f. Point A is one of the x-intercepts of f, while C is the x-intercepts of g. Point B is the y-intercepts of g.



20.1 Determine the area of ΔAOD.

20.2 For which values of p will $-(x-p)^2 - 4(x-p) = 2^x - 6$ have two real negative roots?

20.3 For which values of k will $-x^2 - 4x = 2^x + k$ have two real roots that are opposite in sign?

ooooooooooooooo **END OF QUESTIONS ON ALGEBRAIC FUNCTIONS** ooooooooooooooo

Space for some notes