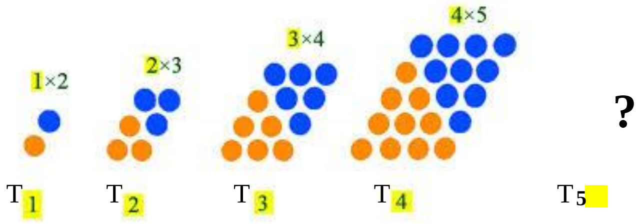
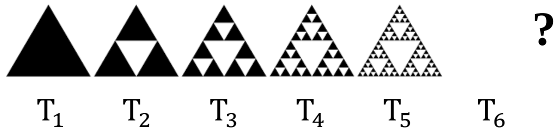
Number Patterns, Series and Sequences



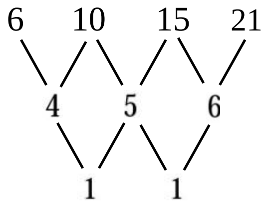


To excel in this topic, a thorough grasp of the following is essential:

- Ability to identify and differentiate between the three main number patterns, viz: linear (i.e. arithmetic), exponential (i.e. geometric) and quadratic series and sequences.
- The n-th term and sum formulas of the above-mentioned three; how to derive these formulas and when and how to apply them.
- The sigma-notation, $\sum_{r=1}^{m}(\ldots.)$, its meaning and application.
- The sum to infinity (S_{∞}) of a geometric series where $-1 < r < 1$ and the concepts of convergence or divergence.

1.1 21 dots.

1.2 The numbers of the triangle sequence are: 6 ; 10 ; 15 ; 21 giving a common 2^{nd} difference of 1, indicating that the sequence is quadratic



$a = \frac{1}{2}(2^{nd}\text{ difference}) = \frac{1}{2}(1) = \frac{1}{2}$

$b = (1^{st}\text{ term of }1^{st}\text{ difference}) - 3a = 4 - 3\left(\frac{1}{2}\right) = 2½$

$c = T_1 - a - b = 6 - \frac{1}{2} - 2½ = 3$

$\therefore T_n = ½n^2 + 2½n + 3$

1.3 $T_{10} = ½(10)^2 + 2½(10) + 3 = 78$ $\therefore$ 10^{th} Δ has 78 dots. Σ

1.4 $T_n = ½n^2 + 2½n + 3 = 990$ $\therefore$ $= n^2 + 5n - 1974 = 0$ $\therefore$ $(n - 42)(n + 47) = 0$
$\therefore$ $n = 42$ or $n = -4$ …n/a
The 42^{nd} term.

2.1 Given the A.S: $-7 - 3 + 1 + \ldots. + 69$ with $a = -7$; $d = 4$ $\therefore$ $T_n = a + (n-1)d$
$\Rightarrow$ $T_n = -7 + 4(n-1) = 4n - 11$ Now, $69 = a + (n-1)d$ $\Rightarrow$ $69 = 4n - 11$
$\therefore n = 20$ i.e. 20 terms.

2.2 $\sum_{n=1}^{20}(4n - 11) = 620$

3.1 a) $\sum_{i=0}^{14}(4i+11) \Rightarrow$ determining the sum of 15 terms where $T_1 = 11,\ T_1 = 15$ and $T_3 = 19$

(**Note**: With the sigma-notation, it is always wise to calculate the first 3 terms as shown on the right.

$d = 4$: $i = 0$: $T_1 = a = 11$; $i = 1$: $T_2 = 15$; $i = 2$: $T_3 = 19$

$\therefore$ We have an AS with $a = 11$ and $d = 4$

$\therefore\ S_{15} = \frac{n}{2}[2a + (n-1)d] = \frac{15}{2}[2(11) + (15-1)4]$

$\therefore\ S_{15} = \frac{15}{2}(78) = 585$

b) $\sum_{n=1}^{\infty}\frac{1}{4}(2)^{n-1} \Rightarrow$ determining the sum to infinity where $T_1 = \frac{1}{4}$ and $r = 2$ (see calculations ↓)

Not possible, as $r = 2$, this geometric series will not converge, but diverge. For convergence, $-1 < r < 1$

$r = 2$: $n = 1$: $T_1 = a = 0{,}25$; $n = 2$: $T_2 = 0{,}5$; $n = 3$: $T_3 = 1$

3.2 Given $\sum_{n=1}^{m}3(2)^{1-n} = 5.8125$, $\Rightarrow$ solve for m where $S_{GS} = 5{,}8125$ (see calculations for 1st 3 terms)

$S_m = \frac{a(1-r^m)}{1-r} = 5{,}8125 \quad \therefore \frac{3(1-(\frac{1}{2})^m)}{1-\frac{1}{2}} = 5{,}8125$

$r = 0{,}5$: $n = 1$: $T_1 = a = 3$; $n = 2$: $T_2 = 1{,}5$; $n = 3$: $T_3 = 0{,}75$

$\therefore 1 - \left(\frac{1}{2}\right)^m = \frac{31}{32} \Rightarrow \left(\frac{1}{2}\right)^m = 1 - \frac{31}{32} = \frac{1}{32}$ i.e. $\left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^5$

$\therefore$ m = 5

OR …

(If one had noticed at the beginning that the sum of the 1st 3 terms = 5,25 i.e. very close to 5,8125, and then generating another one or two more terms, we would see $S_5 = 5{,}8125$, thereby saving oneself all the above calculations , and above all, saving a lot of time… an important factor when writing exams.)

4.1 Number pattern for rods: 2 ; 4 ; 6 ; …. and number pattern for balls: 1 ; 4 ; 9 ; ….

4.2 T_{10} (rods) = 20 and T_{10} (balls) = 100

4.3.1 Rods: $a = 2,\ d = 2 \quad \therefore T_n = a + (n-1)d = 2 + 2(n-1) = 2n \quad \therefore T_n = 2n$

4.3.2 Balls: $T_n = n^2$ ….(an easy identifiable quadratic sequence)

5.1 A GS (Geometric Series) with $a = 10$ and $r = 1{,}1$ $\left(r = \frac{T_2}{T_1} = \frac{11}{10} = 1{,}1\right)$ and $T_n = ar^{n-1}$

$\therefore\ T_8 = 10(1{,}1)^{8-1} = 19{,}487171 \approx 19{,}49$ km.

5.2 $S_8 = \frac{a(r^n-1)}{r-1} = \frac{10((1{,}1)^n-1)}{1{,}1-1} = 114{,}358881 \approx 114{,}36$ km.

6.1 $T_5 = a + 4d = 0$ ....❶ and $T_{14} = a + 13d = -36$ ....❷

$\therefore$ ❷ − ❶: $\therefore\ 9d = -36\ \therefore\ d = -4$

$\therefore\ T_1 = 16$

6.2 $T_{23} + T_{23-p} = -96 \Rightarrow 16 + 22(-4) + 16 + (22-p)(-4) = -96$

$\therefore\ 4p = 48\ \therefore\ p = 12$

7.1 Given the infinite G.S. $-4\ ;\ -\frac{8}{3}\ ;\ -\frac{16}{9}\ ;\ -\ \ldots.$ Now, $r = \frac{T_2}{T_1} = \frac{-\frac{8}{3}}{-4} = \frac{2}{3}$

and since $-1 < \frac{2}{3} < 1 \Rightarrow S_\infty$ exists as it complies with the requirement for convergence.

7.2 $S_\infty = \frac{a}{1-r} = \frac{-4}{1-0,\dot{6}} = -12$

8.1 $d = T_2 - T_1 = T_3 - T_2\ \therefore\ 3p - (2p+14) = (p+7) - 3p\ \therefore 3p - 2p - p + 3p = 7 + 14$

$\therefore\ 3p = 21\ \therefore\ p = 7$

8.2 $T_1 = 2p + 14 = 2(7) + 14 = 28 = a$ and $T_2 = 3p = 3(7) = 21\ \therefore\ d = T_2 - T_1$

$= 21 - 28 = -7.$ And $S_{38} = \frac{n}{2}[2a + (n-1)d] = 19[2(28) + 37(-7)] = -3857$

9. $T_1 = (\text{vol. of 1st pyramid}) = \frac{1}{3}(A_{\text{base}})\ \text{x}\ \perp\text{h}) = \frac{1}{3}(9\text{x}9)(27) = 729\ cm^3$

$T_2 = (\text{vol. of 2nd pyramid}) = \frac{1}{3}(27)(9) = 81\ cm^3$ ....(base and height diminish by a third)

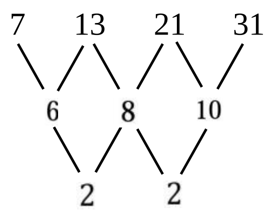
$\therefore\ r = \frac{T_2}{T_1} = \frac{81}{729}\ \ \therefore r = \frac{1}{9}$

$\therefore\ S_\infty = \frac{a}{1-r} = \frac{729}{1-0,\dot{1}} = \frac{729}{0,\dot{8}} = 820{,}125\ cm^3$

10. Simplifying each term of the given sequence $\frac{2^3-1}{1}\ ;\ \frac{3^3-1}{2}\ ;\ \frac{4^3-1}{3}\ ;\ \frac{5^3-1}{4}\ ;\ \ldots.$

we arrive at: $7\ ;13\ ;21\ ;\ 31\ \ldots$

The simplified terms of the sequence have a common 2nd difference of 2, confirming that it is quadratic.



$a = \frac{1}{2}(2^{\text{nd}}\text{ difference}) = \frac{1}{2}(2) = 1$

$b = (1^{\text{st}}\text{ term of }1^{\text{st}}\text{ difference}) - 3a = 6 - 3(1) = 3$

$c = T_1 - a - b = 7 - 1 - 3 = 3$

$\therefore\ T_n = n^2 + 3n + 3$

11. For the given geometric series $9+6+4+\cdots$ $\quad r=\frac{T_2}{T_1}=\frac{6}{9}$ $\quad \therefore r=\frac{2}{3}$

$S_n=\frac{a(1-r^n)}{1-r}=\frac{9\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}$ $\quad \therefore \frac{9\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}>25 \Rightarrow 1-\left(\frac{2}{3}\right)^n>\frac{25}{27}$ $\quad \therefore \left(\frac{2}{3}\right)^n<\frac{2}{27}$

$\therefore n\log\left(\frac{2}{3}\right)<\log\left(\frac{2}{27}\right)$ $\quad \therefore n> \log_{\left(\frac{2}{3}\right)}\left(\frac{2}{27}\right)$ $\quad \therefore n>6{,}419\ldots$ (**Note**: since $\log\left(\frac{2}{3}\right)$ is negative, the inequality sign must change around again).

i.e. 7 terms would be the smallest number of terms for $\text{Sum}_{GS}>25$.

12.1 $T_4=48=ar^3$ ....❶ and $T_7=384$ ....❷

❷ ÷ ❶ : $\frac{384}{48}=\frac{ar^6}{ar^3}$ $\quad \therefore r^3=8$ $\quad \therefore r=2$. Substituting $r=2$ into ❶ : $8a=48$

$\therefore a=6$.

12.2 $T_1=-\frac{3}{4}=a$ ....❶ and $T_4=\frac{2}{9}=ar^3$ ....❷ Substituting $a=-\frac{3}{4}$ into ❷ : $\frac{2}{9}=-\frac{3}{4}r^3$

$\therefore r^3=-\frac{8}{27}$ $\quad \therefore r=-\frac{2}{3}$

Hence, $x=T_2=\left(-\frac{3}{4}\right)\left(-\frac{2}{3}\right)=\frac{1}{2}$ and , $y=T_3=\left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)=-\frac{1}{3}$

13.1 The given sequence $x\,;0\,;3\,;y\,;15$ contains two unknowns, which we can solve by setting up two equations out of the 2nd differences, (which in a quadratic number pattern, are always equal).

Sequence:	x		0		3		y		15
1st difference:		$-x$		3		$y-3$		$15-y$	
2nd difference:			$3+x$		$y-6$		$18-2y$		

Setting 2nd differences equal (last two give): $18-2y=y-6$ $\quad \therefore 3y=24$ $\quad \therefore y=8$

And 1st two give (with $y=8$): $3+x=8-6$ $\quad \therefore x=-1$

13.2 Substituting the values for x and y, we see that 2nd diff. = 2

$a=\frac{1}{2}(2^{\text{nd}}\text{ difference})=\frac{1}{2}(2)=1$

$b=(1^{\text{st}}\text{ term of }1^{\text{st}}\text{ difference})-3a=1-3(1)=-2$

$c=T_1-a-b=-1-(1)-(-2)=0$

$\therefore T_n=n^2-2n$

13.3 $T_n=n^2-2n=224$ $\quad \therefore n^2-2n-224=0 \Rightarrow (n-16)(n+14)=0$

$\therefore n=16$ or $n=-14$ ....n/a

The 16th term.

14. Two sequences are involved here. (i) Success rate: $25\,;28\,;31\,;\ldots.$ An AS with $d = 3$ and (ii) Length of kicks: $25\,;30\,;\ 36\,;\ \ldots.$ Quadratic sequence with 2nd diff. = $\frac{1}{2}$

14.1 $T_{4(AS)} = 31 + 3 = 34$ $\therefore$ 34 successful kicks (out of 50) at goal posts.

14.2 $T_{4(QS)} = 36 + 7 = 43$ $\therefore$ 43 metres in 4th week.

14.2 In week 5, the week of the match (game), his average length kicking to the posts will be $43 + 8 = 51\ m$ – he will be attempting a 48 m kick – and his success at post will be $34 + 3 = 37$ i.e. $\left(\frac{37}{50}\right)\text{x}100 = 74\%$. Thus a 74% chance that he will be successful.

15. $\text{Vol}_{\text{sphere}} = \frac{4}{3}\pi r^3$ and $\text{V}_{\text{1st sphere}} = 36\pi \Rightarrow \frac{4}{3}\pi r^3 = 36\pi \quad \therefore r^3 = 27 \quad \therefore r = 3$ cm

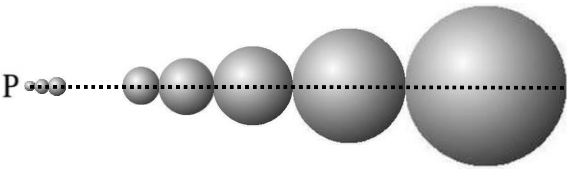
$\text{V}_{\text{2nd sphere}} = \left(\frac{27}{8}\right).36\pi = \frac{243}{2}\pi \Rightarrow \frac{4}{3}\pi r^3 = \frac{243}{2}\pi \quad \therefore r^3 = \frac{729}{8} \therefore r = \frac{9}{2} = 4{,}5$ cm

$\Rightarrow \text{Diameter}_{\text{1st sphere}} = 6$ cm, and

$\text{Diameter}_{\text{2nd sphere}} = 9$ cm

Clearly a GS with ratio ($\mathbf{r}$) $= \frac{T_2}{T_1} = \frac{9}{6} = \frac{3}{2}$

and first term, $a = 6$



Now, $\text{S}_\text{n} = \frac{a(r^n-1)}{r-1} = \frac{6((1{,}5)^n-1)}{1{,}5-1} = \frac{57513}{128} \quad \therefore (1{,}5)^n = \frac{19171}{512} + 1 \quad \therefore (1{,}5)^n = \frac{19683}{512}$

$\therefore n\log\left(\frac{3}{2}\right) = \log\left(\frac{19683}{512}\right) \quad \therefore n = \log_{\left(\frac{3}{2}\right)}\left(\frac{19683}{512}\right) \quad \therefore n = 9.$ There are 9 spheres.

16.1 $\text{T}_1 = 15 \quad \therefore \text{T}_2 = (0{,}8)15 = 12\ m$

16.2 A GS with $a = 15$ and $r = 0{,}8$

$\text{T}_n = ar^{n-1} \Rightarrow 15.(0{,}8)^{n-1} > 6$

$\therefore (0{,}8)^{n-1} > \frac{2}{5}$ (or 0,4)

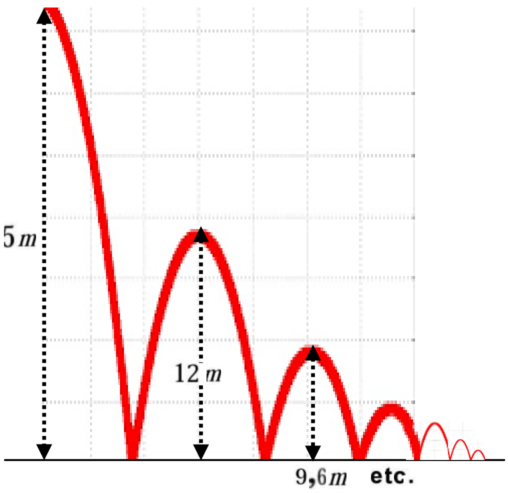
$\therefore (n-1)\log 0{,}8 > \log 0{,}4$

$\therefore n - 1 < \log_{0{,}8} 0{,}4$ i.e $n - 1 < 4{,}106\ldots$

$\therefore$ n $< 5{,}613\ldots \ \therefore$ 5 times.

OR

(By generating the next few terms:
$\text{T}_3 = 0.8(12) = 9{,}6;\ \text{T}_4 = 0{,}8(9{,}6) = 7{,}68$
$\text{T}_5 = 0.8(7.68) = 6.144;\ \text{T}_6 = 0{,}8(6.144)$
$= 4{,}91..$ $\therefore$ 5 times.)



16.3 $\text{S}_\infty = \frac{a}{1-r} = \frac{15}{1-0{,}8} = \frac{15}{0{,}2} = 75$ m. But ball bounces 'up' and ' down', except for the 1st 15 m

$\therefore \text{S}_\infty = 2(75) - 15 = 135$ metres.

(Alternatively, start with $\text{T}_1 = 12$, multiply by 2 and add the 1st 'drop' of 15 m
$\text{S}_\infty = \frac{a}{1-r} = \frac{12}{1-0{,}8} = \frac{12}{0{,}2} = 60$ m. $\therefore \text{S}_\infty = 2(60) + 15 = 135$ metres.)

17. $T_{1\,(AS)} = T_{1\,(GS)} = a = 2$. Now, with respect to the AS: $T_3 = a + (n-1)d = 2 + 2d$
$= T_{2\,(GS)}$ and $T_{11} = a + (n-1)d = 2 + 10d = T_{3\,(GS)}$.
$\therefore$ the first three term of GS: $2\ ;\ 2 + 2d\ ;\ 2 + 10d$
$\Rightarrow \frac{2+2d}{2} = \frac{2+10d}{2+2d}$ …(from from $\frac{T_2}{T_1} = \frac{T_3}{T_2} = r$ for a GS) $\therefore (2+2d)(2+2d) = 2(2+10d)$
$\therefore 4 + 8d + 4d^2 = 4 + 20d \quad \therefore 4d^2 - 12d = 0 \quad \therefore 4d(d-3) = 0$ i.e. $d = 3$ or $d = 0$ ..n/a

17.1 The 1st three terms for the GS: 2 ; 8 ; 32

17.2 $S_{n\,(AS)} = \frac{n}{2}[2a + (n-1)d] \quad \therefore S_6 = \frac{6}{2}[2(2) + (6-1)(3)] = 57$

18.1 $\text{Shape}_1 = 6\ ;\ \text{Shape}_2 = 10\ ;\ \text{Shape}_3 = 14 \quad \therefore$ An AS with $a = 6\ ;\ d = 4$
$T_n = a + (n-1)d = 6 + (n-1)(4) = 4n + 2$
Now, $526 = 4n + 2 \quad \therefore n = 131$. $\therefore$ The 131st term would have 526 matches.

18.2.1 Given the quadratic sequence: 298 ; 259 ; 222 ; 187 ; …… and $T_n = an^2 + bn + 339$
For $n = 1$: $T_1 = 298 = a(1)^2 + b(1) + 339 \quad \therefore a + b = -41$ ……❶
For $n = 2$: $T_2 = 259 = a(2)^2 + b(2) + 339 \quad \therefore 4a + 2b = -80$ ……❷
❶ x 2 : $2a + 2b = -82$ ……❸ and ❷ − ❸ : $2a = 2 \quad \therefore a = 1$
and $b = -42$ ….(from either ❶ or ❷)
$\therefore T_n = n^2 - 42n + 339$

18.2.2 $T_n = n^2 - 42n + 239$

19.1 Losing a minute an hour $\Rightarrow$ 1 min. in 60 min. $T_1 = 60 = a\ ;\ T_2 = \left(\frac{59}{60}\right)60 = 59$
$T_3 = \left(\frac{59}{60}\right)59 = 58{,}016\ldots$ A GS therefore, with $r = \frac{59}{60}$
$\therefore S_\infty = \frac{a}{1-r} = \frac{60}{1-\frac{59}{60}} = \frac{60}{\frac{1}{60}} = \frac{60\text{x}60}{1} = 3600$ minutes …or
$= 60$ hours …or
$= \frac{60}{24} = 2{,}5$ days

19.2 $108\ \text{km/h} = \frac{108000}{3600}\ \text{m/s} = 30$ m/s. $\therefore T_1 = 30 = a\ ;\ T_2 = (0{,}7)30 = 21\ ;$ … etc.
A GS with $a = 30$ and $r = 0{,}7 \quad \therefore S_\infty = \frac{a}{1-r} = \frac{30}{1-0{,}7} = \frac{30}{0{,}3} = 100$
Vehcile will come to a halt 100 m further after applying brakes.

20. $S_8 = -4{,}5 \Rightarrow \frac{8}{2}[2a + 7d] = -4{,}5 \quad \therefore 8a + 28d = -4{,}5$ ……….❶
$T_2 + T_8 = -1 \Rightarrow (a + d) + (a + 7d) = -1 \quad \therefore 2a + 8d = -1$ ….❷
❷x4: $8a + 32d = -4$ ….❸ $\therefore$ ❸ − ❶ : $4d = 0{,}5$ i.e. $d = 0{,}125$ and $a = -1$
$\therefore T_9 = a + 8d = -1 + 8(0{,}125) = 1 - 1 = 0$
$\therefore$ the 9th term is 0.

21.1 a) Cubes: Pattern $_1 = 3\text{x}9 = 27$; Pattern $_2 = 4\text{x}16 = 64$; Pattern $_3 = 5\text{x}25 = 125$

b) Pattern $_4$ will have $6x36 = 216$ $\therefore$ 216 cubes.

c) From the above values, the sequence is: 27 ; 64 ; 125 ; 216 ; … or: 3^3; 4^3; 5^3; 6^3; ...
Clearly a straight forward and easy cubic sequence. $\therefore$ $\text{T}_\text{n} = (n+2)^3$

21.2 a) Each cube has six sides and each side of a specific cube has an equal number of squares.
$\therefore$ Squares: Pattern $_1 = 6\text{x}9 = 54$; Pattern $_2 = 6\text{x}16 = 96$; Pattern $_3 = 6\text{x}25 = 150$

b) Pattern $_4$ will have $6x36 = 216$ $\therefore$ 216 squares. I

c) From the above values, the sequence is: 54 ; 96 ; 150 ; 216 ; … Possibly quadratic.
$\therefore$ Investigate whether a constant 2nd difference? ...Yes

42 54 66

12 12

$a = \frac{1}{2}$ (2nd difference) $= \frac{1}{2}(12) = 6$

$b =$ (1st term of 1st difference) $- \ 3a = 42 - 3(6) = 24$

$\text{c} = \text{T}_1 - \text{a} - \text{b} = 54 - 6 - 24 = 24$

$\therefore$ $T_n = 6n^2 + 24n + 24$

22. Given to solve for m where $\sum_{n=1}^{m} 8(2)^{n-1} < 400$. Generating the 1st three terms, we have:

$r = 2$: $n = 1$: $\text{T}_1 = a = 8$; $n = 2$: $\text{T}_2 = 16$; $n = 3$: $\text{T}_3 = 32$

$\therefore$ A GS with $a = 8$ and $r = 2$

$\text{S}_m = \frac{a(r^m - 1)}{r-1}$ $\therefore \text{S}_m < 400 \Rightarrow \frac{8(2^m - 1)}{2-1} < 400$

$\therefore$ $2^m - 1 < 50$ i.e $2^m < 51$

$\therefore m\log 2 = \log 51$ $\therefore m = \log_2 51$ $\therefore m < 5{,}6724$ ...

$\therefore$ Largest term is T_5, i.e. the 5th term.

(Verification: $8 + 16 + 32 + 64 + 128 = 248$, which is < 400. $\text{T}_6 = 256 \Rightarrow \text{S}_6 = 504$)

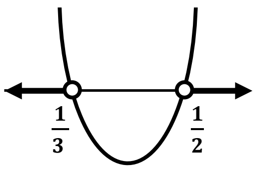
23. (**Note**: For all series and sequences, an important characteristic or property that is often overlooked, is the following: $\text{S}_{n-1} + \text{T}_n = \text{S}_n$ or stated differently: $T_n = S_n - S_{n-1}$)

$\therefore \text{T}_n = \text{S}_n - \text{S}_{n-1}$ $\therefore \text{T}_6 = \text{S}_6 - \ \text{S}_5 = 152 - 90 = 62$

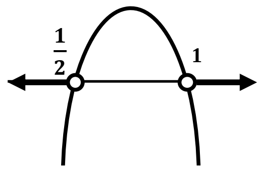
24. (**Note**: Normally, this type of question is quite straight forward and shouldn't pose much of a problem to students. However, in this particular case, the variable is in the denominator (which is seldom the case), and students fall into the trap of multiplying throughout with a variable across an inequality − in order with equations, but not with inequalities. An easy trap to get caught in, mainly due to the paucity of such questions in text books).

24   24.1 Now, in the given geometric series, where $r = \frac{T_3}{T_2} = \frac{T_2}{T_1} = \frac{x}{2x-1}$ and where convergence $\Rightarrow -1 < r < 1$ i.e. $-1 < \frac{x}{2x-1} < 1$ .... a three part inequality with the variable in the denominator and must be handled in two parts, viz. (i) $-1 < \frac{x}{2x-1}$ and (ii) $\frac{x}{2x-1} < 1$ and the two solutions combined at the end as the final answer.

$$-1 < \frac{x}{2x-1}$$
$$0 < \frac{x}{2x-1} + 1$$
$$0 < \frac{x+1(2x-1)}{2x-1}$$
$$0 < \frac{3x-1}{2x-1}$$
$$\therefore\ x < \frac{1}{3} \text{ or } x > \frac{1}{2}$$



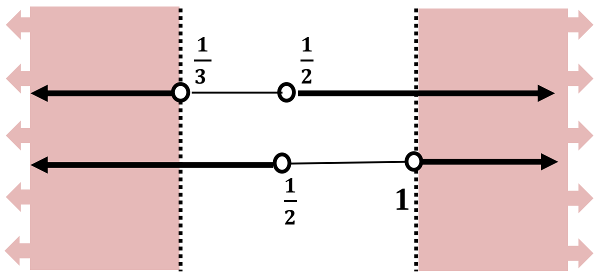
$$\frac{x}{2x-1} < 1$$
$$\frac{x}{2x-1} - 1 < 0$$
$$\frac{x-1(2x-1)}{2x-1} < 0$$
$$\frac{-x+1}{2x-1}$$
$$\therefore\ x < \frac{1}{2} \text{ or } x > 1$$



And now we combine the two parts, giving solutions that are common for both sets (a sketch here is helpful)

Solution of LHS:

Solution of RHS:



The combined solution is: $x < \frac{1}{3}$ or $x > 1$ ….(area shaded in pink)

25. (Interpret and simplify the three sigma-notation terms separately and then recombine the equatio and solve for m).

i) $\sum_{r=1}^{m} 4\left(\frac{1}{2}\right)^r$ $r = 0{,}5 \left\{ \begin{array}{l} n=1:\ T_1 = a = 2 \\ n=2:\ T_2 = 1 \\ n=3:\ T_3 = 0{,}5 \end{array} \right]$ $\Rightarrow S_m = \frac{a(1-r^m)}{1-r}$ $\therefore$ $\frac{2(1-0{,}5^m)}{1-0{,}5} = 4(1-0{,}5^m)$

ii) $\sum_{p=1}^{\infty} \frac{5}{8}\left(\frac{1}{3}\right)^{p-1}$ $\Rightarrow$ A GS with $T_1 = a = \frac{5}{8}$; $r = \frac{1}{3}$ and $S_\infty = \frac{a}{1-r} = \frac{5}{8} \div \left(1 - \frac{1}{3}\right) = \frac{15}{16}$

iii) $\sum_{k=1}^{12} \frac{1}{2}(7-k)$ $\Rightarrow$ An AS with $T_1 = a = 3,\ T_2 = 2{,}5$ and $T_3 = 2$ $\therefore d = -0{,}5$

$\therefore S_{12} = 6[2(3) + 11(-0{,}5)] = 3$

Now $\sum_{r=1}^{m} 4\left(\frac{1}{2}\right)^r = \sum_{p=1}^{\infty} \frac{5}{8}\left(\frac{1}{3}\right)^{p-1} + \sum_{k=1}^{12} \frac{1}{2}(7-k)$ $\Rightarrow 4(1-0{,}5^m) = \frac{15}{16} + 3 = \frac{63}{16}$

i.e. $4(1-0{,}5^m) = \frac{63}{16}$ $\therefore 1 - 0{,}5^m = \frac{63}{64}$ $\therefore \left(\frac{1}{2}\right)^m = 1 - \frac{63}{64}$ $\left(\frac{1}{2}\right)^m = \frac{1}{64}$ $\therefore m = 6$

26.1 $\sum_{k=1}^{14} (\sin 30^0 + \cos 60^0)^{k-1} \Rightarrow \sum_{k=1}^{14} (0{,}5 + 0{,}5)^{14} = 1^{14} = 1$

26.2 $\sum_{p=1}^{\infty} (0{,}1)^p \Rightarrow$ a $= 0{,}1$; $r = \frac{T_2}{T_1} = \frac{0{,}01}{0{,}1} = 0{,}1$ and $S_\infty = \frac{a}{1-r} = \frac{0{,}1}{1-0{,}1} = \frac{0{,}1}{0{,}9} = \frac{1}{9} = 0{,}\dot{1}$

27.1 Generating the 1st 3 terms from $T_1 = 2$, given that $T_{k+1} = \frac{1}{2}T_k \Rightarrow T_2 = \frac{1}{2}T_1 = \frac{1}{2}(2) = 1$ and $\therefore T_3 = \frac{1}{2}T_2 = \frac{1}{2}$, implicating that sequence is geometric …(a common ratio of $r = \frac{1}{2}$)

27.2 From $T_n = ar^{n-1}$ for a GS $\Rightarrow$ $T_k = 2\left(\frac{1}{2}\right)^{n-1}$ **OR** $T_k = \left(\frac{1}{2}\right)^{n-2}$ **OR** $T_k = 2^{2-n}$

28.1 $T_4 = \frac{-7}{125}$

28.2 Numerators: An AS with $a = 2$ & $d = -3$ $\therefore T_n = a + (n-1) = 2 + (n-1)(-3)$
$\therefore T_{n\,(numer.)} = 5 - 3n$. Denominators: A GS with $a = 1$ & $r = 5$ $\therefore T_{n(denom.)} = ar^{n-1}$
$\therefore T_n = \frac{5-3n}{5^{n-1}}$

28.3 $T_{500} = \frac{5-3(500)}{5^{500-1}} = \frac{-1495}{5^{499}}$

28.4 $5 - 3n < -59$ $\therefore$ $-3n < -64$ $\therefore$ $n > 21{,}\dot{3}$ $\therefore$ $n = 22$. i.e. the 22nd term.

29.1 An AS with $a = -3$ & $d = 8$ $\therefore T_k = a + (k-1) = -3 + (k-1)(8) = 8k - 11$
$\therefore T_k = 8k - 11$

29.2 $\sum_{k=1}^{n} (8k - 11)$

29.3 $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[2(-3) + (n-1)(8)] = \frac{n}{2}(8n - 14) = 4n^2 - 7n$

29.4.1 $P_6 == -10 - 3 + 5 + 13 + 21 + 21$

29.4.2 $P_{160} = -10 + S_{159\,(initial\ seq.)} = -10 + 4(159)^2 - 7(159) = 100\ 001$

30.1 Pattern no 3 has 10 blocks and Pattern no 5 has 35 blocks.

30.2 1 ; 3 ; 6 ; 10 ; 15 ; …

30.3 Bottom layer is a quadratic sequence, namely:

1 3 6 10 15
2 3 4 5
1 1 1

$a = \frac{1}{2}$ (2nd difference) $= \frac{1}{2}(1) = \frac{1}{2}$

$b =$ (1st term of 1st difference) $- 3a$

$= 2 - 3\left(\frac{1}{2}\right) = \frac{1}{2}$

$c = T_1 - a - b = 1 - \frac{1}{2} - \frac{1}{2} = 0$

$\therefore T_n = \frac{1}{2}n^2 + \frac{1}{2}n$

30.4 Given $T_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ as the n-th term for number of blocks per pattern.

Now, 3x(Bottom layer sequence) = Total no. of blocks sequence

$\Rightarrow 3\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \quad \therefore 9n^2 + 9n = n^3 + 3n^2 + 2n$

$\therefore n^3 - 6n^2 - 7n = 0 \quad \therefore n(n^2 - 6n - 7) = 0 \quad \therefore n = 0$ or $n = -1$ or $n = 7$

0 and -1 are not applicable. $\therefore$ In the 7th pattern.

31.1 $T_4 = 125$

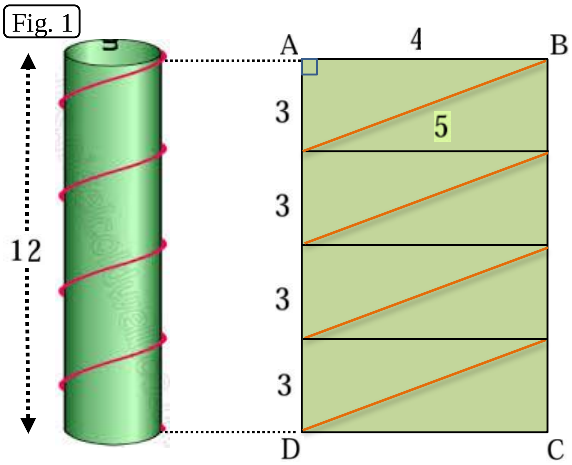
31.2 Sequence is geometric with $a = 1$ and $r = 5 \quad \therefore T_n = ar^{n-1} = 5^{n-1}$

31.3 $S_{10} = \frac{a(r^n - 1)}{r - 1} = \frac{1(5^{10} - 1)}{5 - 1} = 2\,441\,406$ letters.

32.1 Rectangle ABCD represents the curved surface of the cylinder when cut open to form a flat surface, indicating that the 4 windings of the string are the sum of the diagonals, each equalling 5 cm…(Pyth.)

$\therefore$ Length of string in Pattern 1 = $4x5 = 20$ cm.

Fig. 1



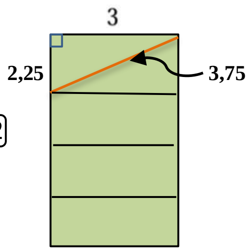
32.2 The rectangle in Fig. 2 represents the opened curved surface of the cylinder of Pattern 2, with circumf. = 0,75x4 = 3 cm and height = 0,75x12 = 9 cm, making each side of the 4 smaller rectangles = 2,25 cm.

$\therefore$ Each diagonal = 3,75 cm (Pyth)

$\therefore$ Length of string in Pattern 2 = 4x3,75 = 15 cm.

$\therefore r = \frac{T_2}{T_1} = \frac{15}{20} = 0{,}75$

Fig. 2



32.3 $\therefore S_\infty = \frac{a}{1-r} = \frac{20}{1-0{,}75} = \frac{20}{0.25} = 80$ cm

oooooooooooooo **END OF ANSWERS on Number Patterns, Series & Sequences** oooooooooooooo