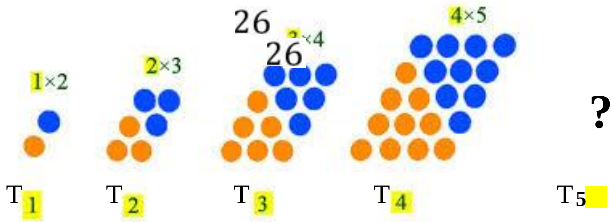
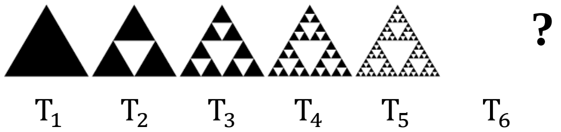
Number Patterns, Series and Sequences

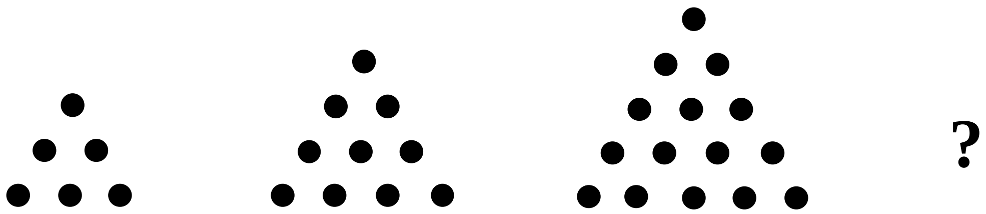




To excel in this topic, a thorough grasp of the following is essential:

- Ability to identify and differentiate between the three main number patterns, viz: linear (i.e. arithmetic), exponential (i.e. geometric) and quadratic series and sequences.
- The n-th term and sum formulas of the above-mentioned three; how to derive these formulas and when and how to apply them.
- The sigma-notation, $\sum$ (….), its meaning and application.
- The sum to infinity (S_∞) of a geometric series where $-1 < r < 1$ and the concepts of convergence or divergence.

1. Consider the following triangle number-pattern:



Pattern no 1 **Pattern no 2** **Pattern no 3** **Pattern no 4**

1.1 How many dots will Pattern no 4 have?

1.2 Determine the general term for the number of dots in the n-th triangle.

1.3 How many dots will there be in the 10^{th} triangle?

1.4 Which term (pattern) will be made up of 990 dots?

2. Given the following arithmetic series: $-7-3+1+\ldots.+69=620$

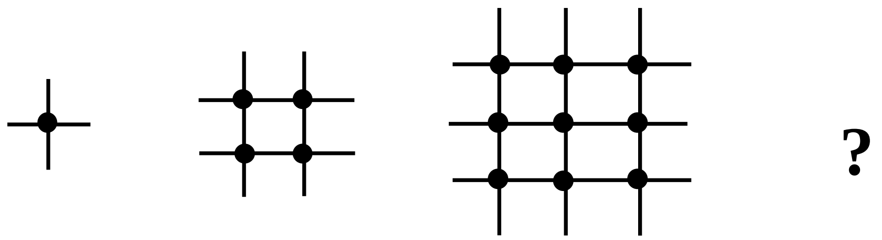
2.1 How many terms are there in the series?

2.2 Write the given series in sigma-notation.

3.1 Evaluate: a) $\sum_{i=0}^{14}(4i+11)$ b) $\sum_{n=1}^{\infty}\frac{1}{4}(2)^{n-1}$

3.2 If $\sum_{n=1}^{m}3(2)^{1-n}=5.8125$, determine the value of m.

4. Consider the following pattern composed of balls and rods:



Pattern no 1 **Pattern no 2** **Pattern no 3** **Pattern no 4** **etc….**

4.1 Give the number of rods and number of balls needed to compose pattern number 4.

4.2 If the above pattern continues to expand in the fashion indicated above, how many balls and how many rods are there in the tenth pattern ?

4.3 Hence, or otherwise, determine the general term for:

4.3.1 the number of rods in the n-th term.

4.3.2 the number of balls in the n-th term.

5. In the first week of training, a student cycles 10 km. Thereafter, the distance he covers each week is 10% more than that of the previous week.

5.1 Determine the distance cycled by the student in the eighth week.

5.2 Determine the total distance cycled by the student in the first 8 weeks.

6. In an arithmetic sequence the fifth term has a value of 0 and the fourteenth term has a value of –36.

6.1 Calculate T_1

6.2 Find the value of p if $T_{23}+T_{23-p}=-96$

7. $-4\,;\,-\frac{8}{3}\,;\,-\frac{16}{9}\,;\,-\,\ldots.$ is an infinite geometric series.

7.1 Explain why the series converges.

7.2 Find the sum to infinity.

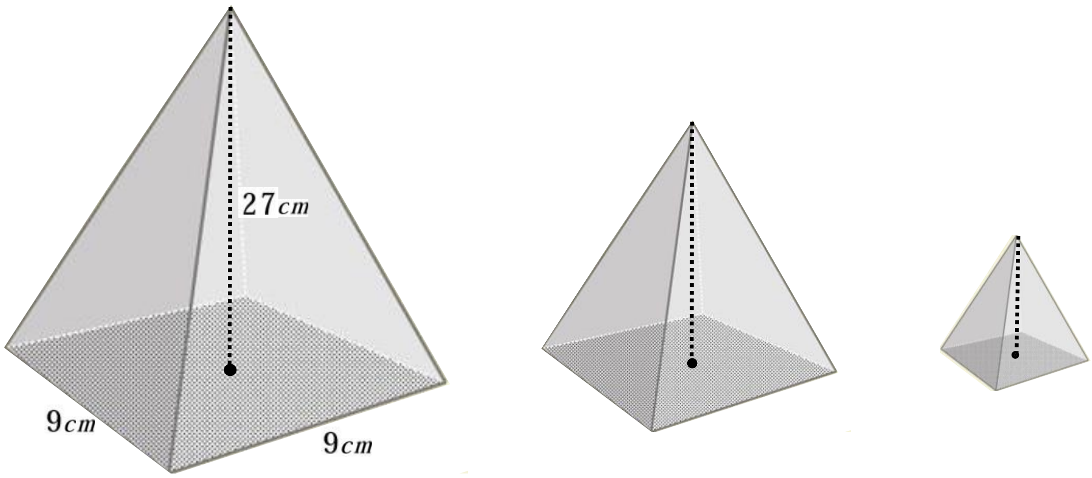
8. An arithmetic sequence is given as: $(2p+14)\,;3p\,;(p+7)\,;\ldots.$

8.1 Determine p.

8.2 Hence determine the sum of the first 38 terms.

9. A solid right pyramid with square base has a perpendicular height of 27 cm. The sides of the base are 9 cm in length. This pyramid is replicated under the following conditions:

- The area of the base and the perpendicular height of each replicate is one third of the previous one.



9.1 Determine the total volume of all the pyramids, if this replication process continues ad infinitum, i.e. indefinitely. (Volume of a pyramid $=\frac{1}{3}$(Area of base) x $\perp$ height).

10. The following sequence is given: $\frac{2^3-1}{1}\,;\,\frac{3^3-1}{2}\,;\,\frac{4^3-1}{3}\,;\,\frac{5^3-1}{4}\,;\,\ldots.$

Given that the above sequence is quadratic, determine its n-th term.

11. The sum of the first n terms of a geometric series $9+6+4+\cdots$ is greater than 25. Determine the smallest value of n for which this happens.

12.1 The 4th term of a geometric series is 48 and the 7th term is 384. Determine the 1st term and the common ratio.

12.2 Determine the value of x and y so that the sequence $-\frac{3}{4}\,;x\,;y\,;\frac{2}{9}$ will be a geometric sequence.

13. If $x\,;0\,;3\,;y\,;15$ form the first five terms of a quadratic sequence, determine:

13.1 the values of x and y.

13.2 the n-th (or general) term.

13.2 the term that has a value of 224.

14. Giuseppe plays rugby for the Red Bulls and is the team's main penalty kicker. He practices his kicking five days a week with the aim of further improving his skills. During the first week of the season he kicks 50 balls of which 25 are successful. He also gradually increases the length of his kicks to the posts starting with 25 m in the first week. During the second week, he kicked an average length (to the posts) of 30 metres and had 28 successes. During the third week the kicking lengths averaged 36 m of which 31 were successful.

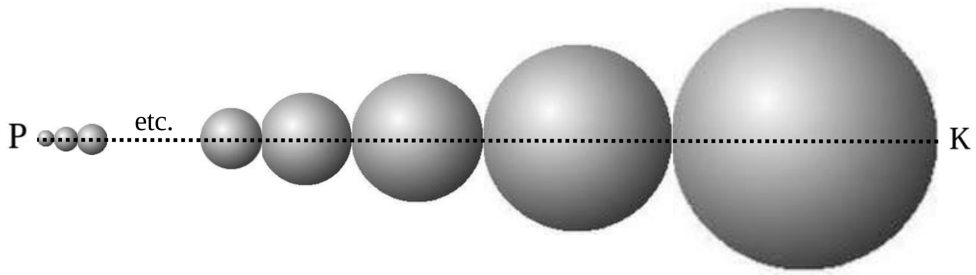
14.1 Assuming that this tendency continues during the 4th week, what will:

14.1.1 his average successful kicks at posts be?

14.1.2 the average length of each kick be?

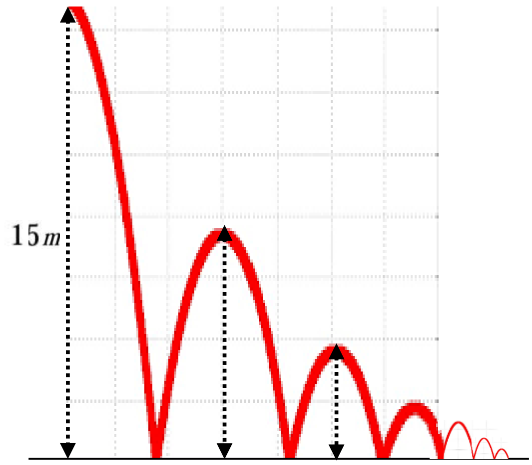
14.2 In a rugby match at the end of the 5th week, Giuseppe attempts a kick of 48 m to the goal posts. What are his chances that he will succeed with this kick, based on his success rate of the previous week?

15. A number of solid spheres are placed alongside each other in line with their diameters as shown in the sketch. The volume of the smallest sphere (at P) is 36π cm^3 and the volume of every subsequent sphere is $\frac{27}{8}$ greater than that of the previous one.



If the sum of the diameters of all the spheres is the distance PK, and PK $= \frac{57513}{128}$ cm, (≈ 450 cm), determine the total number of spheres in this arrangement

16. A ball is dropped from a height of 15 m above the ground and bounces back every time to a height of 80% of its previous bounce. This action continues indefinitely as depicted in the sketch. Determine:



16.1 the height of the ball's first bounce up.

16.2 how many times it bounces higher than 6 metres.

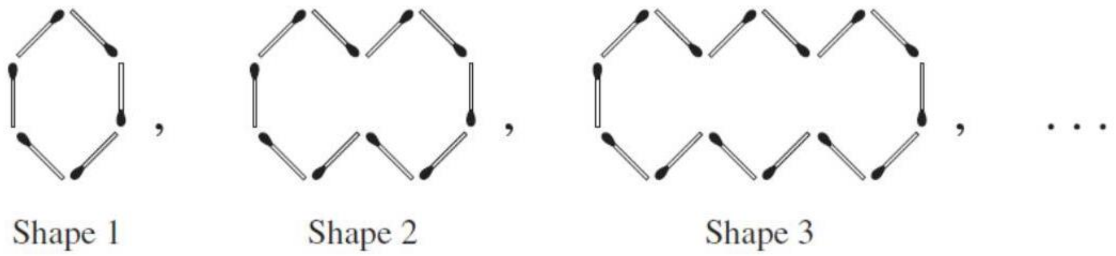
16.3 the total distance that the ball covers before it comes to a stop. (Theoretically though, it never stops bouncing).

17. The 1st, 3rd and 11th terms of an arithmetic sequence are also the first three terms of a geometric sequence. If the 1st term is 2, determine:

17.1 the first three terms of the geometric sequence.

17.2 the sum of the first 6 terms of the arithmetic sequence.

18.1 Consider the three shapes below made of matchsticks:



If the pattern is continued, which shape would use exactly 526 matchsticks?

18.2 Consider the sequence 298 ; 259 ; 222 ; 187 ; 154 ; 123 ; ....
The nth term for the sequence is given by $T_n = an^2 + bn + 339$

18.2.1 Find the values of a and b.

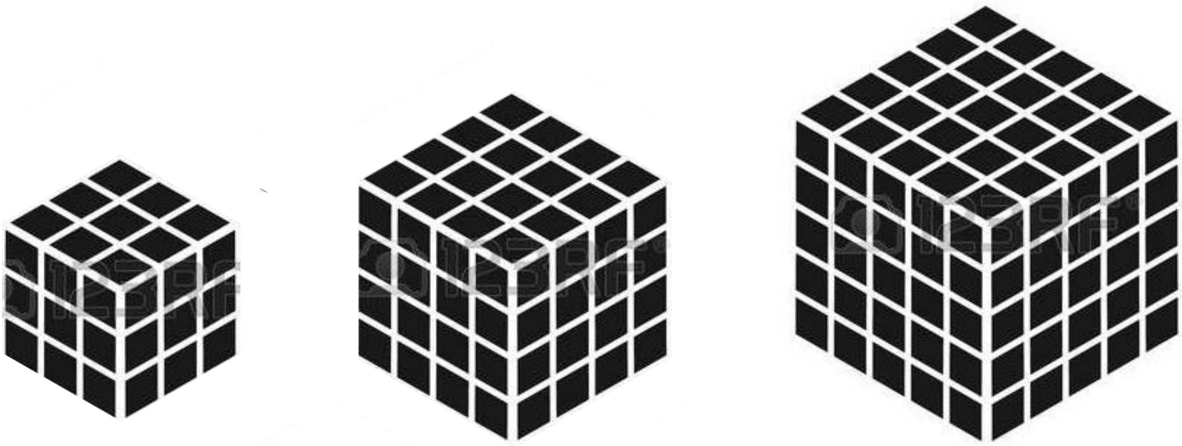
18.2.2 Write down the nth term of the sequence 198; 159; 122; 87; 54; 23; .....

19.1 A Grandfather clock of the old-fashioned mechanical type, starts losing a minute an hour if its not rewound after a few days. How long will it take to stop ticking if this should happen?

19.2 A motor vehcile travelling at 108 km/h will deminish its speed by $\frac{7}{10}$ per second if the brakes are evenly applied for a sudden emergency stop. Determine how far the vehcile will travel before it comes to a halt and how long will this take?

20. The sum of the 1st eight terms of an arithmetic sequence is $-4{,}5$. The sum of the 2nd and 8th term is -1. What will the 9th term be?

21. Consider the following pattern of cubes where each compact cube is made up of a number of smaller cubes and a corresponding number of exposed square surfaces. Pattern 1 is the is the well-known Rubik's cube.



Pattern 1 Pattern 2 Pattern 3

21.1 Determine:
a) the number of cubes for Pattern 1, 2, and 3
c) the number of cubes in the 4th pattern
d) the n-th term for the above sequence of compact cubes.

21.2 Determine:
a) the number of exposed square surfaces for Pattern 1, 2, and 3
c) the number of exposed square surfaces in the 4th pattern
d) the n-th term for the above sequence of total number of squares per each subsequent cube.

22. Determine the largest value of m for which $\sum_{n=1}^{m} 8(2)^{n-1} < 400$.

23. If in a quadratic series the $S_5 = 90$ and $S_6 = 152$, determine the 6th term of the quadratic series.

24.1 For which of values of x will the following infinite series converge?

$$\frac{x}{2x-1}\ ;\ \left(\frac{x}{2x-1}\right)^2;\ \left(\frac{x}{2x-1}\right)^3;\ \left(\frac{x}{2x-1}\right)^4;\ \ldots.$$

24.2 What happens to the series for any other value of x, other than those determined in the previous question, 24.1?

25. Determine the value of m if:

$$\sum_{r=1}^{m} 4\left(\frac{1}{2}\right)^{r} = \sum_{p=1}^{\infty} \frac{5}{8}\left(\frac{1}{3}\right)^{p-1} + \sum_{k=1}^{12} \frac{1}{2}(7-k)$$

26. Evaluate each of the following: ….

26.1 $\sum_{k=1}^{14}(\sin 30^0 + \cos 60^0)^{k-1}$

26.2 $\sum_{p=1}(0{,}1)^p$

27. A sequence is defined as follows: $T_{k+1} = \frac{1}{2}T_k$, where $T_1 = 2$.

27.1 Is the sequenc arithmetic or geometric?

27.2 Determine the general term for T_k.

28. The following sequence has the property that the row of numerators form an arithmetic sequence and the row of denominators form a geometric sequence: $\frac{2}{1}\ ;\ \frac{-1}{5}\ ;\ \frac{-4}{25}\ ; \ldots$

28.1 Write down the FOURTH term of the sequence.

28.2 Determine a formula for the n-th term.

28.3 Determine the 500th term of the sequence.

28.4 Which term will be the 1st term with a numerator smaller than -59?

29. Consider the series: $S_n = -3 + 5 + 13 + 21 + \ldots$ to n terms.

29.1 Determine the general term of the series in the form $T_k = bk + c$.

29.2 Write S_n in sigma notation.

29.3 Show that $S_n = 4n^2 - 7n$.

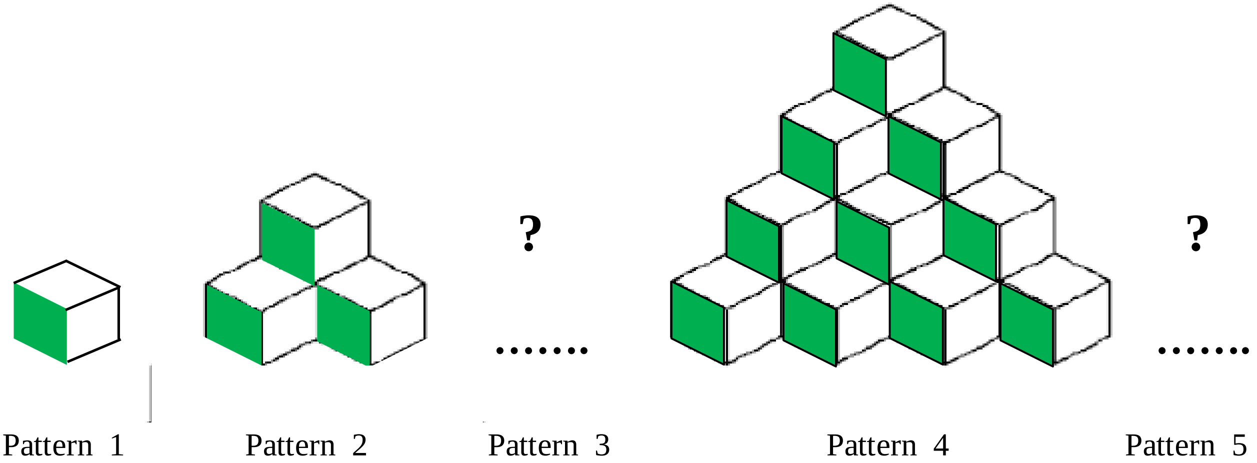
29.4 Another sequence is defined as:

$P_1 = -10$
$P_2 = -10 - 3$
$P_3 = -10 - 3 + 5$
$P_4 = -10 - 3 + 5 + 13$
$P_5 = -10 - 3 + 5 + 13 + 21$

29.4.1 Write down a numerical expression for P_6

29.4.2 Calculate the <u>value</u> of P_{160} .

30. Wooden toy blocks are stacked in a corner of a room forming the patterns as indicated below.



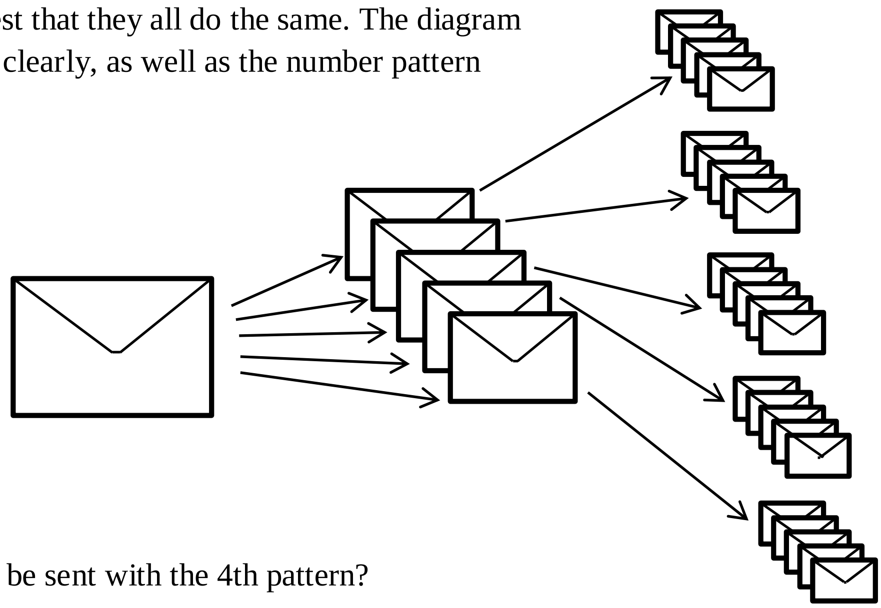
30.1 Write down the number of blocks required to form pattern 3 and pattern 5 respectively.

30.2 Write down the sequence for the number of blocks in the bottom layer of each of the five patterns.

30.3 Determine the n-th term for the number of blocks in the bottom layer of each pattern.

30.4 If $T_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ is the n-th term's formula for the total number of blocks per pattern, determine algebraically in which pattern will a third of the total number of blocks be contained in its bottom layer.

31. A student begins a chain letter by sending five letters to five other students with the request that they all do the same. The diagram illustrates the situation clearly, as well as the number pattern that has been created.



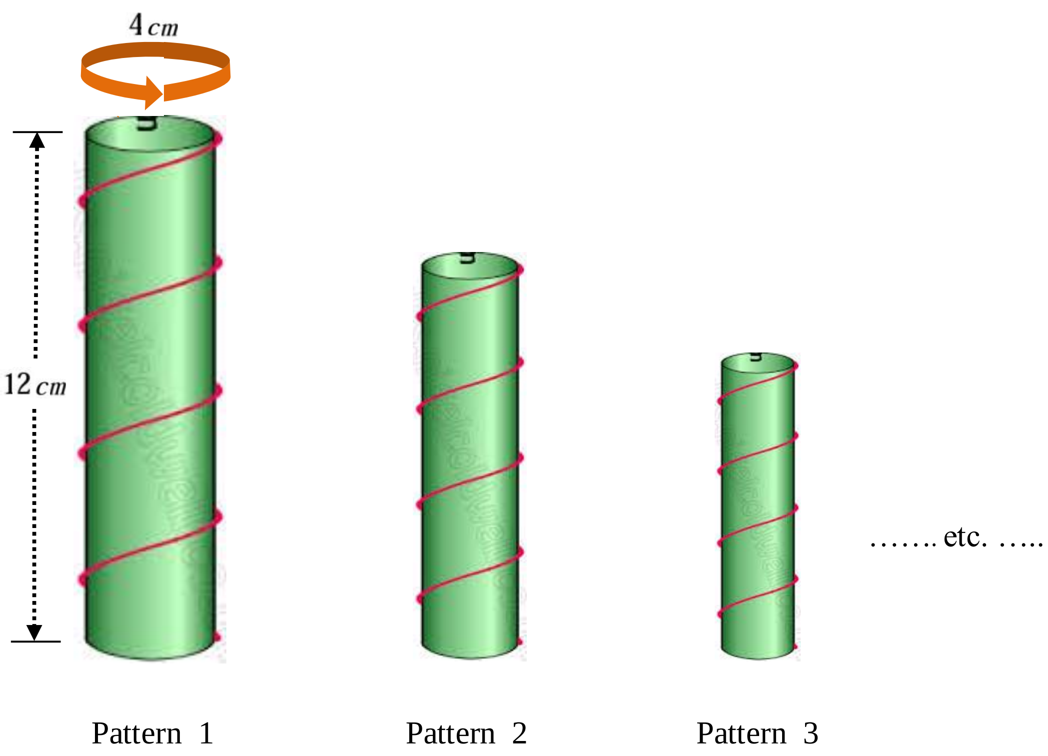
31.1 How many letters will be sent with the 4th pattern?

31.2 Determine the n-th term of this sequence.

31.3 How many letters will have been sent all together at the end of the 10th pattern, assuming that nobody breaks the chain?

32. A string is wound around a cylinder that has a height of 12 cm and a circumference of 4cm. If the string is given four winds around the cylinder, starting at (say) the top on the right and ending at the bottom on the right, directly below the starting point (see sketch),

32.1 Determine the length of the piece of string in Pattern 1.



32.2 The above scenario of a string wound around a cylinder is replicated an infinite number of times under the following condition: Both the circumference and height of each new replicate is $\frac{7,5}{10}$ -ths that of the previous one. Show that the lengths of the strings of these replicates also decreases by 25% , i.e. each string's length is 0,75 that of the previous one.

32.3 Determine S_{∞} of the lengths of all the strings wound around each cylinder.

oooooooooooooO **END OF QUESTIONS on Number Patterns, Series & Sequences** Oooooooooooooo

Space for some notes